Test 2, Linear Algebra

Fall 2017, Dr. Adam Graham-Squire

Name: Key

I pledge that I have neither given nor received any unauthorized assistance on this exam.

(*signature*)

DIRECTIONS

1. Show all of your work. A correct answer with insufficient work will lose points.
2. Read each question carefully and make sure you answer the the question that is asked. If the question asks for an explanation, make sure you give one.
3. Clearly indicate your answer by putting a box around it.
4. Calculators are allowed on this exam. There are certain calculations that you must do by hand and show your work, however. If you do these solely on the calculator you will lose points.
5. Make sure you sign the pledge.
6. For the Technology section of the test (questions 1-4), all questions are required, though you only need to do *one* of the applied questions (Question 4). For the No Technology section, the first 2 questions (#5 and 6) are required. Of the last five questions (numbers 7 through 11), I will drop your lowest score. Thus you can choose to only do 4 of the last five questions, if you wish.
7. Number of questions = 11. Total Points = 50.

1. (5 points) Find the inverse of the matrix $\left[\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 9 & 5 \end{array}\right]$.

You should show the work to do the calculation by hand, but you can check your answer on a calculator if you wish.

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right] \text{ inverse is } \left[\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & B^{-1} \end{array}\right]$$

$B = \begin{bmatrix} 4 & 2 \\ 9 & 5 \end{bmatrix}$ $\Rightarrow$ $ad-bc = 20-18 = 2$

$B^{-1} = \frac{1}{2}\begin{bmatrix} 5 & -2 \\ -9 & 4 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \; -2R_2+R_3 \to R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right] \Leftarrow -R_3+R_1 \to R_1$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 2.5 & -1 \\ -4.5 & 2 \end{bmatrix}$$

Inverse is
$$\begin{bmatrix} 2 & 2 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2.5 & -1 \\ 0 & 0 & 0 & -4.5 & 2 \end{bmatrix}$$

2. (5 points) Construct the following matrices, and explain why your matrices do what you say they do.

   (a) Construct a 3×2 matrix A such that $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

   (b) Construct a 3×2 matrix B such that $B\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

1.5

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ has a nontrivial solution b/c it has a free variable. E.g. $A\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$ but $\begin{bmatrix} 0 \\ 4 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

✓

1.5

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ has only the trivial solution because it has no free variables.

✓

$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$ is $\vec{0}$ only when $x_1 = x_2 = 0$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ which is trivial.

3. (6 points) Calculate the determinants. Either show your work and do the calculation by hand, or give an explanation for how you can find the answer with no calculation. You can check your work on a calculator/computer if you wish.

4 (a) $\det\begin{bmatrix} 2 & 0 & 1 & -2 \\ 0 & 3 & 1 & -1 \\ 4 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \cancel{-0} + 3\det\begin{bmatrix} 2 & 1 & -2 \\ 4 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} + \cancel{0[\,]}$

$\begin{matrix} 2 & 1 & -2 & 2 & 1 \\ 4 & 0 & 1 & 4 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{matrix}$

$\begin{matrix} 2 & 1 & -2 & 2 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{matrix}$

Check Stan

$= 3(0+1-8-0-2-4) - 2(2-1-0-(-2)-(-2)-0)$

$= 3(-13) - 2(5)$

$= -39 - 10 = \boxed{-49}$

2 (b) $\det\begin{bmatrix} 1 & -2 & 3 & -4 \\ 3 & 17 & 82 & -12 \\ 5 & -10 & 15 & -20 \\ -2 & 4 & -31 & 8 \end{bmatrix}$

Row 3 is 5 times Row 1

$\Rightarrow$ Scalar multiples

$\Rightarrow$ Linearly dependent

$\Rightarrow$ Not invertible (IMT)

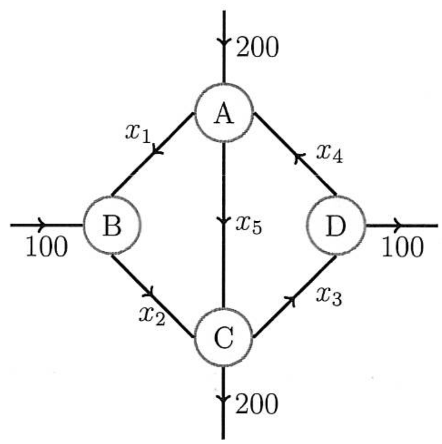
$\Rightarrow$ det $= \boxed{0}$

4. (5 points) Choose *one* of the problems below to solve. You should use linear algebra and some form of technology to find your answer. Note that if you use a computer, you should ONLY use the approved linear algebra calculation websites (i.e. you should not be googling "how to I balance a chemistry equation with linear algebra"). Make sure to explain your work!

- Chemistry question: Find appropriate weights to balance the chemical equation:

$$KNO_3 + C_{12}H_{22}O_{11} \rightarrow N_2 + CO_2 + H_2O + K_2CO_3$$

- Network flow question: Consider the given network, assume all flows are nonnegative. Find the general flow pattern of the network, and use it to explain what the minimum and maximum values are for x_4.



- Input-output economy question: The Dominic economy consists of the Cats, Legos, and Origami sectors.

  The total output of the Cats sector is 35% to Legos, 45% to Origami, and 20% to itself.

  The total output of the Legos sector is 15% to Origami, 75% to Cats, and 10% to itself.

  The total output of the Origami sector is 40% to Legos, 55% to Cats, and 5% to itself.

  If possible, find equilibrium prices that make each sector's income match its expenditures.

The next page is left blank for you to solve whichever problem you choose.

$$\begin{array}{c} K \\ N \\ O \\ C \\ H \end{array} \quad \text{Matrix is } \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -2 \\ 1 & 0 & -2 & 0 & 0 & 0 \\ 3 & 11 & 0 & -2 & -1 & -3 \\ 0 & 12 & 0 & -1 & 0 & -1 \\ 0 & 22 & 0 & 0 & -2 & 0 \end{bmatrix} \xrightarrow[\text{to}]{\text{computer reduces}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & -5/24 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -3/2 \\ 0 & 0 & 0 & 0 & 1 & -55/24 \end{bmatrix}$$

Let $x_6 = 24 \Rightarrow$

$$x_1 = 48$$
$$x_2 = 5$$
$$x_3 = 24$$
$$x_4 = 36$$
$$x_5 = 55$$

$$\Rightarrow 48\,KNO_3 + 5\,C_{12}H_{22}O_{11} = 24\,N_2 + 36\,CO_2 + 55\,H_2O + 24\,K_2CO_3$$

B

Network Flow

A: $x_4 + 200 = x_1 + x_5$

B: $x_1 + 100 = x_2$

C: $x_2 + x_5 = x_3 + 200$

D: $x_3 = x_4 + 100$

$$\Rightarrow \begin{aligned} x_1 \quad\quad\quad\quad - x_4 + x_5 &= 200 \\ x_1 - x_2 \quad\quad\quad\quad\quad\quad &= -100 \\ x_2 - x_3 \quad\quad + x_5 &= 200 \\ x_3 - x_4 \quad\quad &= 100 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 1 & 200 \\ 1 & -1 & 0 & 0 & 0 & -100 \\ 0 & 1 & -1 & 0 & 1 & 200 \\ 0 & 0 & 1 & -1 & 0 & 100 \end{array}\right]$$

reduces to

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 1 & 200 \\ 0 & 1 & 0 & -1 & 1 & 300 \\ 0 & 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ x_4 and x_5 are free

$x_3 = 100 + x_4$ $\quad\quad$ $x_3 - x_4 = 100$

$x_2 + x_5 = 300 + x_4$

$x_1 + x_5 = 200 + x_4$

x_4 can be as little as zero, and has no maximum b/c x_5 is also free.

Economy:

	Cats	Lego	Or	
	20	75	55	Cats
	35	10	40	Lego
	45	15	5	Or

$$\Rightarrow \begin{aligned} 0.2C + 0.75L + 0.55R &= C \\ 0.35C + 0.10L + 0.40R &= L \\ 0.45C + 0.15L + 0.05R &= R \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} -0.8 & 0.75 & 0.55 & 0 \\ 0.35 & -0.9 & 0.40 & 0 \\ 0.45 & 0.15 & -0.95 & 0 \end{array}\right]$$

reduces to $\left[\begin{array}{ccc|c} 1 & 0 & -106/61 & 0 \\ 0 & 1 & -205/183 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\Rightarrow C = \frac{106}{61}R \qquad L = \frac{205}{183}R$$

$\boxed{\text{Let } R = 183 \Rightarrow C = 318,\ L = 205}$

No Technology section

Name Key

5. (4 points) True or False: If true, briefly explain why. If false, explain why or give a counterexample.

   (i) If v_1, v_2, v_3 are in $\mathbb{R}^3$ and v_3 is <u>not</u> a linear combination of v_1 and v_2 then the set $\{v_1, v_2, v_3\}$ is linearly independent.

   ✓ <u>False</u> v_1 and v_2 could be linearly dependent. ✓

   E.g. $\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix}2\\0\\0\end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{v_3}$

   (ii) For any square matrix A and scalar c, $\det(cA) = c \det A$.

   ✓ <u>False!</u> Should be $\det(cA) = c^n(\det A)$ where A is $n \times n$ matrix ✓

   E.g. $A = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ $c = 2$

   $\det\begin{bmatrix}2&0\\0&2\end{bmatrix} = 4$ $\quad 2 \det A = 2 \cdot 1 = 2$

   (iii) If A is a 3×4 matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ must be onto $\mathbb{R}^3$.

   ✓ <u>False!</u> A could have only two pivots

   e.g. $A = \begin{bmatrix}1&0&*&*\\0&1&*&*\\0&0&0&0\end{bmatrix}$ is <u>not</u> onto $\mathbb{R}^3$ ✓

   (iv) If an $n \times n$ matrix A is invertible, then the columns of A^T are linearly independent.

   ✓ <u>True.</u> A invertible $\Rightarrow A^T$ invertible (by IMT)

   $\Rightarrow$ columns of A^T are lin. indep (IMT) ✓

0.5 pts free

6. (5 points) Show that the transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ 4 + x_2 \\ 7x_1 + 3x_2 \\ 0 \end{bmatrix}$$

is <u>not</u> linear.

$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2(0) - 0 \\ 4 + 0 \\ 7(0) + 3(0) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} \neq \vec{0}$$

So <u>cannot</u> be linear.

Of the remaining 5 questions, your lowest score will be dropped

7. (5 points) (a) Show that if the columns of B are linearly dependent, then so are the columns of AB.

Let $B = [b_1 \cdots b_n]$. Then lin. dep. means there are $c_1 \ldots c_n$ scalars such that

$$B \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots + c_n \vec{b}_n = \vec{0}$$

2.5 Then $(AB)\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = A(\vec{0}) = \vec{0}$

So columns of AB are lin. dependent by definition as well.

+1.5 for example

(b) Is it true that if the columns of B are linearly independent then so are the columns of AB? Explain or give a counterexample.

2.5 False. A could be linearly dependent. E.g.

If $A = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & 0 \end{bmatrix}$ the zero matrix, then

$AB = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & 0 \end{bmatrix}$ is not linearly independent columns.

Even better answer: Suppose B is 3×3 and lin. indep. if A is 2×3, then AB is 2×3 and must be lin. dep. b/c more columns than rows.

8. (5 points) Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$.

(a) Construct a 2×3 matrix C using only 1, 0 and/or -1 as entries, such that $CA = I_2$ (I_2 is the 2×2 identity matrix).

(b) Is it possible to find a 2×3 matrix C such that $AC = I_3$? Why or why not?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

many answers

2.5

(b) If $CA = I$ and $AC = I$ then A must be invertible by def. Since A is 3×2, not square, it cannot be invertible.

2.5

or $\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \underbrace{\begin{bmatrix} a & b & c \\ d-a & e-b & f-c \\ -d & -e & -f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$

means $a = 1$, $f = -1$

$b = c = d = e = 0$

and $e - b = 1$, but

$e - b = 0$ (since both are 0)

$\Rightarrow \Leftarrow$ so it can't happen.

or by Question #7, since C is 2×3 it has lin. dep. columns

$\Rightarrow AC$ has lin. dep. columns

$\Rightarrow AC \neq I_3$

9. (5 points) (a) Suppose the columns of an $n \times n$ matrix A are linearly independent. Explain why the columns of A^3 must span $\mathbb{R}^n$.

columns of A lin indep $\Rightarrow$ A ✓ invertible (IMT) ✓

$\Rightarrow \det A \neq 0$ ✓

$\Rightarrow (\det A)^2 \neq 0$

$\Rightarrow \det A^3 \neq 0$ ✓ b/c $(\det A)(\det A)(\det A) = \det(A^3)$

$\Rightarrow A^3$ invertible ✓

$\Rightarrow$ columns of A^3 span $\mathbb{R}^n$ (IMT) ✓

$\to n \geq m$

(b) If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$, what can you say about the relative sizes of m and n? What can you say if T is one-to-one? Explain your reasoning.

$\to m \geq n$

$n > m$

$$\begin{array}{cc} & 4 \\ 2 & \left[\quad\quad\quad \right] \end{array}$$

$\Downarrow$

Can be onto, b/c can have pivot in every row

Cannot be 1-1 b/c no pivot in every column

$n = m$

$\Downarrow$

Can be onto and 1-1 (can pivot in every row & column)

$n < m$

$$\begin{array}{cc} & 2 \\ 4 & \left[\quad \right] \end{array}$$

Cannot be onto b/c not enough pivots to have one in every row, but

can be 1-1 b/c can have pivot in every column (no free variables)

10. (5 points) (a) Compute $\det(B^{40})$ where $B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix}$. (Hint: you don't have a calculator, so you probably cannot actually calculate what the matrix B^{40} is.)

$\det(B^{40}) = (\det B)^{40}$

$= (-1)^{40}$

$= \boxed{1}$

$\det B = \begin{bmatrix} 1 & 3 & 2 & 1 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 4 & 1 & 1 & 4 \end{bmatrix}$

$= 1 + 0 + 0 - 2 - 0 - 0$

$= -1$

2.5

(b) Let A be a 2×2 matrix such that $\det(A^2) = 0$. Does A have to be the zero matrix? If so, explain why. If not, give a counterexample (that is, a nonzero 2×2 matrix such that $\det(A^2) = 0$).

2.5 No. Could have $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (or any other non-invertible matrix). Then $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

so $\det(A^2) = 0$ but

$A \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

11. (5 points) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set, but $\{T(\mathbf{u}), T(\mathbf{v})\}$ is a linearly *dependent* set. Show that $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution. [Hint: Use the fact that $c_1T(\mathbf{u})+c_2T(\mathbf{v}) = \mathbf{0}$ for some scalars c_1 and c_2, not both zero.]

$\{T(\vec{u}), T(\vec{v})\}$ lin. dep $\Rightarrow$ there are $\underbrace{c_1, c_2}_{\text{Not both zero}}$ such that

$$c_1 T(\vec{u}) + c_2 T(\vec{v}) = \vec{0}$$

$$\Rightarrow T(c_1\vec{u}) + T(c_2\vec{v}) = \vec{0}$$

$$\Rightarrow T(c_1\vec{u} + c_2\vec{v}) = \vec{0}$$

Since c_1 and c_2 are not both zero, $c_1\vec{u}+c_2\vec{v} \neq 0$

b/c $\vec{u}$ and $\vec{v}$ are lin. indp. Thus

$T(\text{something} \neq 0) = \vec{0}$ so T has a nontrivial

Solution.

Extra Credit (2 points): Let T be the following transformation on 2×2 matrices: $T(A) = \det(A)$ for all 2×2 matrices A. Is T a linear transformation? Explain why or why not.

No; not at all. Namely, to be Linear must have

$\det(cA) = T(cA) = c\,T(A) = c\det(A)$, and we proved in #5(ii)

that $\det(cA) \neq c \det A$.